MUMBAI UNIVERSITY

SEMESTER 2

APPLIED MATHEMATICS SOLVED PAPER – DECEMBER 2017

N.B:- (1) Question no. 1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1.(a) Evaluate $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx$ **[3]**

Ans : Let **I =** $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx$

Put $x^2 = t$ $=>$ **x =**$\sqrt{t}$ $=>$ $\sqrt{x} = t^{1/4}$

Differentiate w.r.t x,

$\therefore$ **dx =** $\frac{1}{2\sqrt{t}}\, dt$ **lim ⟶ [0,∞]**

$$\therefore \mathrm{I} = \int_0^\infty \frac{e^{-t}}{1} \frac{t^{-3/4}}{2}\, dt$$

$$\therefore \mathrm{I} = \frac{1}{2}\int_0^\infty e^{-t}.\, t^{\frac{1}{4}-1} dt$$

But we know that ,

$$\int_0^\infty e^{-t}.\, t^{n-1} dt = gamma(n)$$

$\boxed{\therefore \mathrm{I} = \frac{1}{2}\Gamma(\frac{1}{4})}$ ……………..{By the definition of gamma fn}

(b) Solve $(D^3 + 1)^2 y = 0$ **[3]**

Ans : $(D^3 + 1)^2 y = 0$

For complementary solution,

f(D) = 0

$$\therefore\ (D^3 + 1)^2 = 0$$

$$\therefore \quad (D^3+1) = 0$$

Roots are : D = -1 , $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$ .. for two times

Roots of given diff. eqn are real and complex.

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c = (c_1+xc_2)e^{-x} + e^{\frac{x}{2}}[(c_3+xc_4)cosx + (c_5+xc_6)sinx]}$$

(c) Solve the ODE $(y+\frac{1}{3}y^3+\frac{1}{2}x^2)\,dx + (x+xy^2)dy = 0$ [3]

Ans : Compare the given diff. eqn with Mdx + Ndy = 0

$\therefore$ M = $(y+\frac{1}{3}y^3+\frac{1}{2}x^2)$ $\therefore$N = $(x+xy^2)$

$\frac{\partial M}{\partial y} = 1+y^2$ $\frac{\partial N}{\partial x} = 1+y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential eqn is exact.

The solution of exact differential eqn is given by,

$$\int Mdx + \int[N - \frac{\partial}{\partial y}Mdx]\,dy = c$$

$$\int Mdx = \int (y+\frac{1}{3}y^3+\frac{1}{2}x^2)\,dx = xy+\frac{x}{3}y^3+\frac{x^3}{6}$$

$$\frac{\partial}{\partial y}\int Mdx = x+xy^2$$

$$\int[N - \frac{\partial}{\partial y}Mdx]\,dy = \int[\,x+xy^2-(x+xy^2)]dy = 0$$

$$\boxed{\therefore \quad xy+\frac{x}{3}y^3+\frac{x^3}{6} = c}$$

(d) Use Taylor's series method to find a solution of $\frac{dy}{dx} = 1 + y^2, y(0) = 0$

At x=0.1 taking h=0.1 correct upto 3 decimal places. [4]

Ans : $\frac{dy}{dx} = 1 + y^2 \qquad x_0 = 0, y_0 = 0$, **h=0.1**

$$y' = 1 + y^2 \qquad y_0' = 1$$

$$y'' = 2yy' \qquad y_0'' = 0$$

$$y''' = 2yy'' + 2y'.y' \qquad y_0''' = 2$$

Taylor's series is given by :

y(0.1) = $y_0 + h.y_0' + \frac{h^2}{2!}y_0'' + \cdots$

= 0 + 0.1(1) + $\frac{0.1\times0.1}{2}(0)$ + $\frac{0.1\times0.1\times0.1}{6}(2)$

$\boxed{\text{y(0.1)} = 0.10033}$

(e) Given $\int_0^x \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})$ using DUIS find the value of

$\int_0^x \frac{dx}{(x^2+a^2)^2}$ **[4]**

Ans : $\int_0^x \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

Differentiate w.r.t a , taking 'a' as parameter

$\frac{d}{da}\int_0^x \frac{1}{x^2+a^2}\,dx = \frac{d}{da}[\frac{1}{a}\tan^{-1}(\frac{x}{a})]$

Applying D.U.I.S rule,

D.U.I.S rule says that if function and its partial derivative is continuous then we can apply differential operator in the integral operator by converting it into partial derivative taking one parameter fro function.

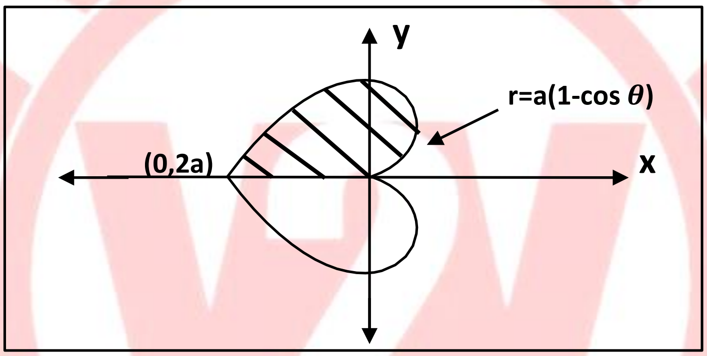
$$\int_0^x \frac{\partial}{\partial a}\frac{1}{x^2+a^2}\,dx = -\frac{1}{a}\tan^{-1}\frac{x}{a}\times\frac{1}{a} + \frac{-x}{a(x^2+a^2)}$$

$$\int_0^x \frac{2a^2}{x^2+a^2}\,dx = -\frac{1}{a}\tan^{-1}\frac{x}{a}\times\frac{1}{a} + \frac{-x}{a(x^2+a^2)}$$

$$\boxed{\int_0^x \frac{dx}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}}$$

(f) Find the perimeter of the curve r=a(1-cos θ) **[4]**

Ans : **Curve : r=a(1-cos θ)**



Perimeter of given curve is ,

$$S = 2\times\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$\frac{dr}{d\theta} = a(\sin\theta) \quad => \quad \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^2\theta$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2[1-2\cos\theta+\cos^2\theta] + a^2\sin^2\theta$$

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}a\,(1-\cos\theta)^{1/2}$$

$$= \sqrt{2}a\sqrt{2}\,\sin\left(\frac{\theta}{2}\right)$$

$$\therefore \quad S = 2\int_0^{\pi}\sqrt{2}a\sqrt{2}\,\sin\left(\frac{\theta}{2}\right)d\theta$$

$$= 4a\int_0^{\pi}\sin\left(\frac{\theta}{2}\right)d\theta$$

$$= 4a\left[-2\cos\left(\frac{\theta}{2}\right)\right]_{0}^{\pi}$$

$$\boxed{\therefore \quad S = 8a}$$

Q.2.(a) Solve $(D^3+D^2+D+1)y = sin^2x$ **[6]**

Ans : $(D^3+D^2+D+1)y = \sin^2 x$

For complementary solution ,

$$f(D) = 0$$

$$\therefore (D^3+D^2+D+1) = 0$$

Roots are : D = -1 , +i , -i

The complementary solution of given diff eqn is ,

$$\boxed{y_c = c_1 cos\, x + c_2 sin\, x + c_3 e^{-x}}$$

For complementary solution ,

$$y_p = \frac{1}{f(D)} X = \frac{1}{(D^3+D^2+D+1)}\ sin^2x = \frac{1}{2(D^3+D^2+D+1)}(1 - cos2x)$$

$$= \frac{1}{2(D^3+D^2+D+1)}\ e^{0x} - \frac{1}{2(D^3+D^2+D+1)} cos\ 2x$$

$$= \frac{1}{2} - \frac{1}{2} \times \frac{1}{-D-4+D+1} cos\ 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{1}{D+1}\ cos\ 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{1}{D+1}\frac{D-1}{D-1}\ cos\ 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{D-1}{D^2-1}\ cos\ 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{-2sin\ 2x - cos\ 2x}{-4-1}\ cos\ 2x$$

$$\boxed{y_p = \frac{1}{2} + \frac{1}{30}.(2sin\ 2x + cos\ 2x)}$$

The general solution of given diff. eqn is given by,

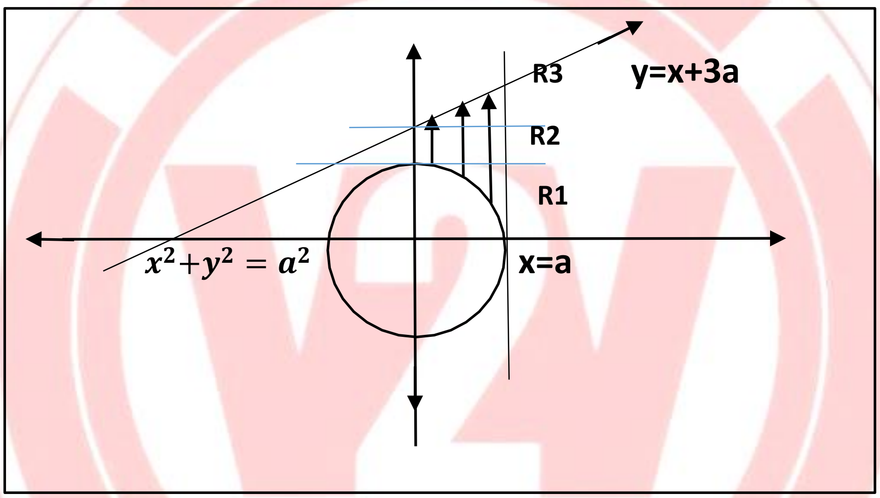
$$\boxed{y_g = y_c + y_p = c_1\cos x + c_2\sin x + c_3e^{-x} + \frac{1}{2} + \frac{1}{30}.(2\sin 2x + \cos 2x)}$$

(b) Change the order of integration $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)dxdy$ **[6]**

Ans : let I = $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)dxdy$

Region of integration is : $\sqrt{a^2-x^2} \le y \le x+3a$

$$0 \le x \le a$$



Intersection of x=a and y=x+3a is (a,4a).

Intersection of x=0 and y=x+3a is (0,3a).

Divide the region into three parts R1,R2 and R3

$\therefore$ **R = R1** $\cup R2 \cup R3$

For region R1 : $\sqrt{a^2-y^2} \le x \le$ a

$$0 \le y \le a$$

For region R2 : $0 \le x \le a$

$$a \le y \le 3a$$

For region R3 : $(y-3a) \le x \le a$

$$3a \le y \le 4a$$

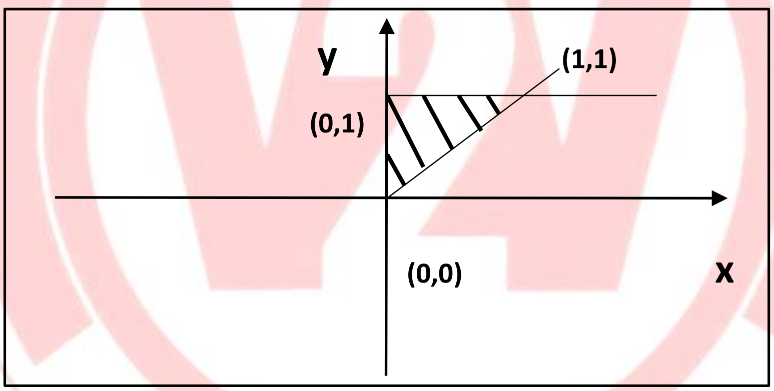
After changing the order of integration fro dydx to dxdy

$$\therefore I = \int_0^a \int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dxdy + \int_a^{3a}\int_0^a f(x,y)\,dxdy + \int_{3a}^{4a}\int_{(y-3a)}^{4a} f(x,y)\,dxdy$$

(c) Evaluate $\int\int \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}}\,dxdy$ **, where R is triangle whose vertices are (0,0),(1,1),(0,1).** **[8]**

Ans : **let** $I = \int\int \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}}\,dxdy$

Region of integration : Triangle whose vertices are (0,0),(1,1),(0,1)



The equation of lines from diagram are : y=1,x=y

$$0 \le x \le y$$

$$0 \le y \le 1$$

$$\therefore I = \int_0^1\int_0^y \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}}\,dxdy$$

$$= \int_0^1\int_0^y \frac{2y^5.x}{\sqrt{(1-y^4)+x^2y^2}}\,dx\,dy$$

$$= \int_0^1 \int_0^y \frac{2y^5.x}{\sqrt{\frac{1-y^4}{y^2}+x^2}} . \frac{1}{y}\ dx\, dy$$

$$= \int_0^1 2y^4 [\sqrt{\frac{1\ \ y^4}{y^2} + x^2}\ ]_0^y\, dy$$

$$= \int_0^1 2y^4\, [\frac{1}{y} - \frac{\sqrt{1-y^4}}{y}]\, dy$$

$$= 2\int_0^1 [\, y^3 - \sqrt{1-y^4}\, .\, y^3]\, dy$$

$$= 2[\frac{y^4}{4} + \frac{1}{4} . \frac{(1-y^4)^{3/2}}{3/2}\ ]_0^1$$

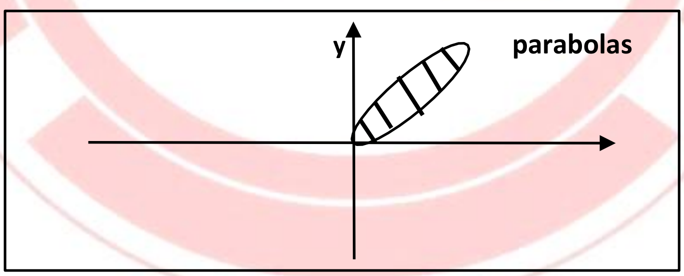
$$= 2[\ \frac{1}{4} - \frac{1}{4} . \frac{2}{3}]$$

$$\boxed{\therefore \mathrm{I} = \frac{1}{6}}$$

Q.3.(a) Find the volume enclosed by the cylinder $y^2 = x$ and $y = x^2$

Cut off by the planes z=0,x+y+z=2. [6]

Ans : The solid is bounded by the parabolas $y^2 = x, x^2 = y$ in the x y plane .



In x-y-z plane x+y+z =2 is top base.

The volume between this curves is given by ,

V = $\int\int z\, dx\, dy = \int\int (2 - x - y)\, dxdy$

From the diagram we can conclude that the intersection point of both

Parabolas are (0,0),(1,1).

$$\therefore V = \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - x - y)\,dx\,dy$$

$$= \int_0^1 \left[2y - xy - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(2\sqrt{x} - x\sqrt{x} - \frac{x}{2}\right) - \left(2x^2 - x^3 - \frac{x^4}{2}\right) \right] dx$$

$$= \left[\frac{4x^{3/2}}{3} - \frac{2x^{\frac{5}{2}}}{5} - \frac{x^2}{4} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1]$$

$$\boxed{\therefore V = \frac{11}{30}}$$

(b) Using Modified Eulers method ,find an approximate value of y At x=0.2 in two step taking h=0.1 and using three iteration Given that $\frac{dy}{dx}$=x+3y , y=1 when x=0. **[6]**

Ans : **(I)** $\frac{dy}{dx}$=x+3y $\quad x_0 = 0, y_0 = 1, h = 0.1$

$y_1^{(0)} = y_0 + h.f(x_0, y_0)$ **= 1+ 0.1(3)=1.3**

$y_1^{n+1} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^n)]$

Iteration	x_1	y_1^n	$x_1 y_1$	y_1^{n+1}
0	0.1	1.3	4	1.35
1	0.1	1.35	4.15	1.3575
2	0.1	1.3575	4.1725	1.3587

$\boxed{y(0.1)=1.3587}$

(II) $x_1 = 0.1, y_1 = 1.3587$

$y_2^0 = 1.77631$

$y_2^{n+1} = y_1 + \frac{h}{2}\,[f(x_2, y_2) + f(x_2, y_2^n)]$

Iteration	x_2	y_2^n	$x_2 y_2$	y_2^{n+1}
0	0.2	1.77631	5.52893	1.8439

1	0.2	1.8439	5.7317	1.8540
2	0.2	1.8540	5.762	1.8556

y(0.2)=1.8556

(c) Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4cos\,(log(1+x))$ **[8]**

Ans : $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4cos\,(log(1+x))$

Put x+1 = v $=>$ $\frac{dv}{dx}=1$

$\frac{dy}{dx}=\frac{dy}{dv}$

The given eqn changes to ,

$$v^2\frac{d^2y}{d_v^2}+v\frac{dy}{dv}+y=4cos\;log\;v$$

Now put log v = z $\therefore$**v=**e^z

$$[D(D-1)+D+1]y=4cos\;z$$

$$\therefore\;(D^2+1)y=4cos\;z$$

For complementary solution ,

$$f(D)=0$$

$$\therefore(D^2+1)=0$$

Roots are : i,-i

The complementary solution of given diff. eqn is ,

$$\boxed{\therefore\;y_c=c_1cos\;z+c_2sin\;z}$$

For particular integral ,

$$y_p=\frac{1}{f(D)}\;X=\frac{1}{(D^2+1)}\;4\;cos\;z=4\frac{z}{2}\;sin\;z=2\text{ z sin z}$$

$$\boxed{\therefore\;y_p=2\;z\;sin\;z}$$

The general solution of given diff. eqn is given by,

$$y_g = y_c + y_p = c_1 \cos z + c_2 \sin z + 2\, z \sin z$$

Resubstitute z and v,

$$\boxed{y_g = c_1 \cos\,[\log(x+1)] + c_2 \sin\,[\log(1+x)] + 2\log(1+x) \sin\,[\log(1+x)]}$$

Q.4.(a) Show that $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\,dx = a\frac{\sqrt{\pi}\,\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}$ **[6]**

Ans : Let I = $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\,dx$

Put $x^3 = a^3 t$ => $x = at^{\frac{1}{3}}$

Diff. w.r.t. x ,

$$dx = \frac{a}{3}\,t^{-2/3}\ \text{dt}$$

Limits becomes $\rightarrow [0, 1]$

$$I = \int_0^1 (t)^{3/2}.(1-t)^{3/2}.t^{-2/3}\frac{a}{3}\,dt$$

$$= \frac{a}{3}\int_0^1 t^{5/6}(1-t)^{3/2}dt$$

$$= \frac{a}{3}\,\beta(\frac{5}{6},\frac{3}{2})$$

$$\boxed{I = a\frac{\sqrt{\pi}\,\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}}$$

…………{ from the definition of beta function}

(b) Solve $(D^2+2)y = e^x \cos x + x^2 e^{3x}$ **[6]**

Ans : $(D^2+2)y = e^x \cos x + x^2 e^{3x}$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 + 2) = 0$$

Roots are : $D = \sqrt{2}i\ , -\sqrt{2}i$

Roots of given diff. eqn are complex.

The complementary solution of given diff. eqn is given by,

$$\boxed{\therefore\ y_c = c_1 cos\,\sqrt{2}x + c_2 sin\,\sqrt{2}x}$$

For particular integral ,

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^2+1}e^x cos\,x + \frac{1}{D^2+1}x^2 e^{3x}$$

$$= e^x\ \frac{1}{(D+1)^2+1}cos\,x + \frac{1}{D^2+1}x^2 e^{3x}$$

$$=\ e^x \frac{1}{D^2+2D+3}cos\,x + e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$=\ e^x\ \frac{1}{2}\ \frac{D-1}{D^2-1}.cosx +\ e^{3x}\frac{1}{D^2+6D+11}x^2$$

$$=\ e^x \frac{1}{4}(sin\,x + cos\,x) + \frac{e^{3x}}{11}[1 + \frac{6D+D^2}{11}]^{-1}x^2$$

$$=\ e^x \frac{1}{4}(sin\,x + cos\,x) + \frac{e^{3x}}{11}[\ 1 - \frac{6D+D^2}{11} + \frac{36D^2}{121} +..]x^2$$

$$\boxed{\therefore\ \ y_p\ \ \ =\ e^x \frac{1}{4}(sin\,x + cos\,x) + \frac{e^{3x}}{11}[\ x^2 - \frac{12x}{11} + \frac{50}{121}]}$$

The general solution of given diff. eqn is,

$$\boxed{y_g = y_c + y_p = c_1 cos\sqrt{2}x + c_2 sin\sqrt{2}x + e^x\frac{1}{4}(sin\,x + cosx) + \frac{e^{3x}}{11}[\ x^2 - \frac{12x}{11} + \frac{50}{121}]}$$

(c) Use polar co ordinates to evaluate $\int\int \frac{(x^2+y^2)^2}{x^2y^2}\ dx\ dy$ over yhe area

Common to circle $x^2 + y^2 = ax\ and\ \ x^2 + y^2 = by, a > b > 0$. [8]

Ans : Let I = $\int\int \frac{(x^2+y^2)^2}{x^2y^2}\, dx\, dy$

Region of integration is : Area common to the circle

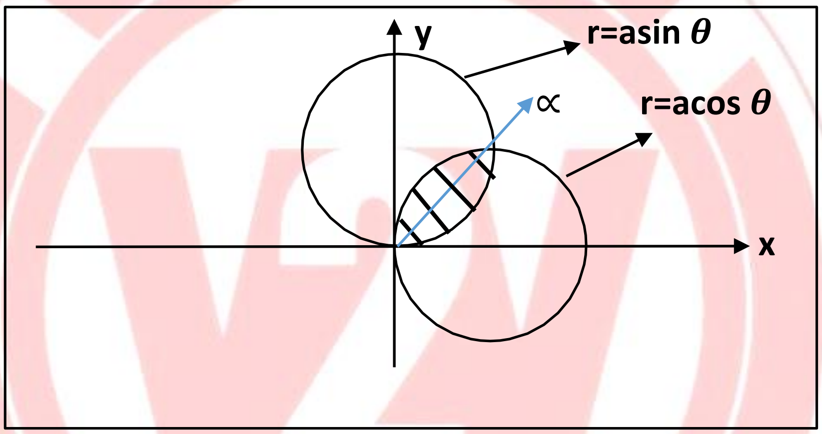
$$x^2+y^2 = ax \text{ and } x^2+y^2 = by$$

To change the Cartesian coordinates to polar coordinates

Put x= r cosθ and y = r sin θ

Circles : r=acos θ *and* r=asin θ

The function becomes : f(x,y)= $\frac{(x^2+y^2)^2}{x^2y^2} = \frac{r^4}{r^4\sin^2\theta\ \cos^2\theta} = \frac{4}{\sin^2 2\theta} = f(r,\theta)$



Intersection of both circles is at angle = $\tan^{-1}\frac{a}{b}$.

Divide the region into two equal halves.

For one region , $0 \le r \le b\sin\theta$

$0 \le \theta \le \alpha$

For another region , $0 \le r \le a\cos\theta$

$\alpha \le \theta \le \frac{\pi}{2}$

$$\therefore I = \int_0^{\alpha}\int_0^{b\sin\theta} \frac{4rdrd\theta}{\sin^2 2\theta} + \int_0^{a\cos\theta}\int_{\alpha}^{\frac{\pi}{2}} \frac{4rdrd\theta}{\sin^2 2\theta}$$

$$\therefore I = \int_0^{\alpha} \frac{4}{\sin^2 2\theta}\left[\frac{r^2}{2}\right]_0^{b\sin\theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{4}{\sin^2 2\theta}\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}b^2\int_0^{\propto} \sec^2\theta d\theta + \frac{a^2}{2}\int_{\alpha}^{\frac{\pi}{2}} \mathrm{cosec}^2\theta d\theta$$

$= \frac{1}{2} b^2 tan \propto + \frac{a^2}{2} cot \propto$

$= \frac{ab}{2} + \frac{ab}{2}$

$\boxed{\therefore I = ab}$

Q.5.(a) Solve $ydx + x(1 - 3x^2y^2)dy = 0$ **[6]**

Ans : $ydx + x(1 - 3x^2y^2)dy = 0$ ……………(1)

Compare the given eqn with Mdx + Ndy=0

$\therefore M = y$ $\therefore N = x(1 - 3x^2y^2)$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 1 - 9x^2y^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence the given diff. eqn is not exact.

But the given diff. eqn is in the form of

yf(xy)dx + xf(xy)dy = 0

Integrating factor = $\frac{1}{Mx-Ny} = \frac{1}{xy-xy+3x^3y^3} = \frac{1}{3x^3y^3}$

Multiply the I.F. to eqn (1),

$\frac{1}{3x^3y^2}$ **dx +** $[\frac{1}{3x^2y^3} - \frac{1}{y}]$**dy = 0**

$\therefore$ $M_1 = \frac{1}{3x^3y^2}$ $N_1 = [\frac{1}{3x^2y^3} - \frac{1}{y}]$

Now this diff. eqn is exact.

The solution of given diff. eqn is given by,

$$\int Mdx + \int [N - \frac{\partial}{\partial y} Mdx]\, dy = c$$

$$\int M_1 dx = \int \frac{1}{3x^3y^2} dx = \frac{-1}{6y^2x^2}$$

$\frac{\partial}{\partial y}\int M_1 dx = \frac{1}{3x^2y^3}$

$\int [\, N_1 - \frac{\partial}{\partial y}\int M_1 dx \,]\, dy = \int [\frac{1}{3x^2y^3} - \frac{1}{y} - \frac{1}{3x^2y^3}]\, dy$

$= \int \frac{-1}{y} dy == -\log y$

$$\boxed{\therefore \frac{-1}{6y^2x^2} - \log y = c}$$

(b) Find the mass of a lamina in the form of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

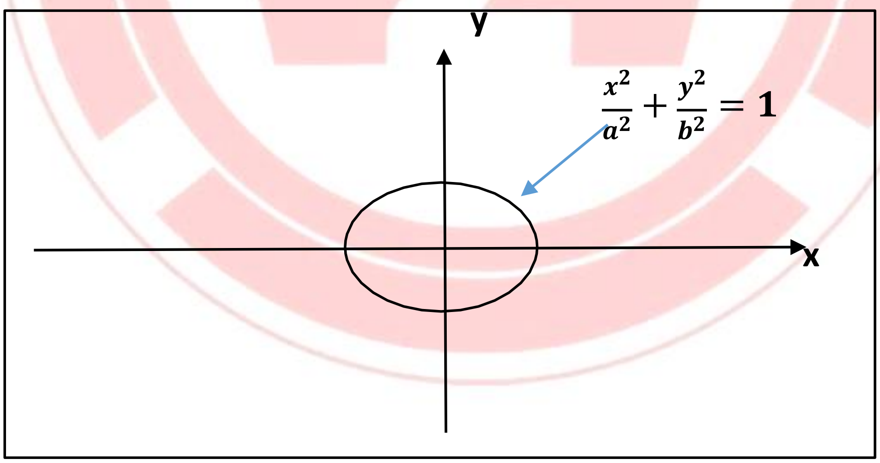
If the density at any point varies as the product of the distance from the

The axes of the ellipse. **[6]**

Ans : Mass of lamina is given by , M = $\int\int r\, dx\, dy$

r is the density function r =k xy

Ellipse eqn is : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$0 \le y \le b\sqrt{a^2 - x^2}/a$$

$$0 \le x \le a$$

$\therefore$ M = $4\int_0^a \int_0^{b\sqrt{a^2-x^2}/a} kxy\, dy\, dx$

$= 4k \int_0^a x.[\frac{y^2}{2}]_0^{b\sqrt{a^2-x^2}/a} dx$

$= 2k \int_0^a x.\frac{b^2}{a^2}(a^2 - x^2)dx$

$= \frac{2kb^2}{a^2}\int_0^a [a^2 x - x^3]dx$

$= \frac{2kb^2}{a^2}[\frac{a^2x^2}{2} - \frac{x^4}{4}]_0^a$

$$\boxed{\therefore M = \frac{ka^2b^2}{2}}$$

(c) Compute the value of $\int_0^{\frac{\pi}{2}} \sqrt{sin\ x + cos\ x} dx$ using (i)Trapezoidal rule (ii)Simpson's (1/3)rd rule (iii)Simpson's (3/8)th rule by dividing into six Subintervals. [8]

Ans : Let I = $\int_0^{\frac{\pi}{2}} \sqrt{sin\ x + cos\ x}\ dx$

Dividing limits into 6 subintervals . n=6

a=0 , b=$\frac{\pi}{2}$ $\therefore$**h=**$\frac{b-a}{n} = \frac{\pi}{12}$

$x_0 = 0$	$x_1 = \pi/12$	$x_2 = 2\pi/12$	$x_3 = 3\pi/12$	$x_4 = 4\pi/12$	$x_5 = 5\pi/12$	$x_6 = 6\pi/12$
$y_0 = 1$	$y_1 = 1.1067$	$y_2 = 1.1688$	$y_3 = 1.1892$	$y_4 = 1.1688$	$y_5 = 1.1067$	$y_6 = 1$

(i) Trapezoidal rule : I = $\frac{h}{2}[X + 2R]$ **-----------------(1)**

$$X = sum\ of\ extreme\ ordinates = 2$$

$$R = sum\ of\ remaining\ ordinates = 5.7402$$

I = $\frac{\pi}{12 \times 2}(2 + 2(5.7402))$ **……………….(from 1)**

$$\boxed{I = 1.7636}$$

(ii) Simpson's $(1/3)^{rd}$ rule :

I = $\frac{h}{3}[\,X + 2E + 4O\,]$ ---------------(2)

$$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 1 + 1 = 2$$

$$E = sum\ of\ even\ base\ ordinates = \ y_2 + y_4 = 2.3376$$

$$O = sum\ of\ odd\ base\ ordinates = \ y_1 + y_3 + y_4 = 3.4026$$

I = $\frac{\pi}{3\times 12}(2 + 2 \times 2.3376 + 4 \times 3.4026)$ ....(from 2)

$\boxed{\text{I = 1.7693}}$

(iii) Simpson's $(3/8)^{th}$ rule :

I = $\frac{3h}{8}[\,X + 2T + 3R\,]$ -------------(3)

$$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 0 + 0.5 = 2$$

$$T = sum\ of\ multiple\ of\ three\ base\ ordinates = \ y_3 = 1.1892$$

$$R = \ sum\ of\ remaining\ ordinates \ = \ y_1 + y_2 + y_4 + y_5 = 4.551$$

$\therefore$ I = $\frac{3\times\pi}{8\times 12}[\,2 + 2 \times 1.1892 + 3 \times 4.551\,]$

$\boxed{\therefore \mathrm{I} = 1.7702}$

Q.6(a)Change the order of Integration and evaluate $\int_0^2 \int_{\sqrt{2y}}^{2} \frac{x^2}{\sqrt{x^4 - 4y^2}}\,dxdy$ **[6]**

Ans : Let I = $\int_0^2 \int_{\sqrt{2y}}^{2} \frac{x^2}{\sqrt{x^4 - 4y^2}}\,dxdy$

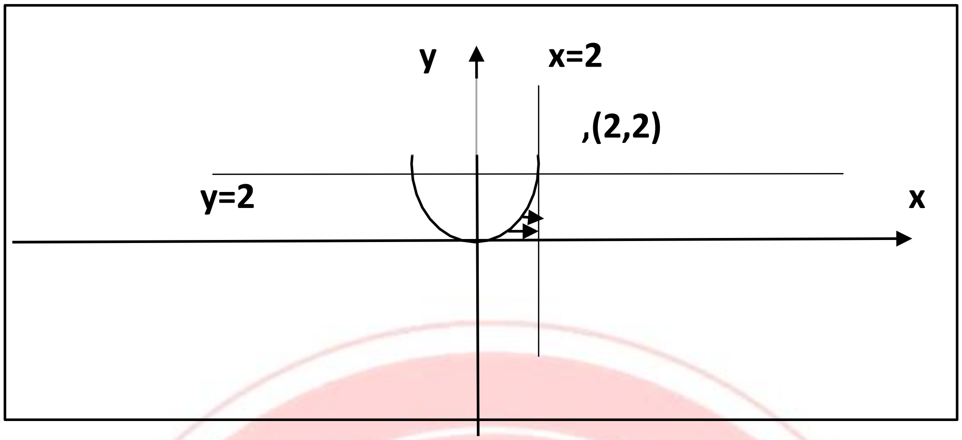
Region of integration : $\sqrt{2y} \le x \le 2$

$$0 \le y \le 2$$

Curves : (i) x = 2 , y = 2 , y = 0 are lines.

(ii) x = $\sqrt{2y}$ => $x^2 = 2y$

Parabola with vertex (0,0) opening in upward direction.



After changing the order of integration:

$$0 \le y \le \frac{x^2}{2}$$

$$0 \le x \le 2$$

$\therefore$ **I** $= \int_0^2 \int_0^{\frac{x^2}{2}} \frac{x^2}{\sqrt{x^4 - 4y^2}}\, dydx$

$= \frac{1}{2} \int_0^2 \int_0^{\frac{x^2}{2}} \frac{x^2}{\sqrt{\frac{x^4}{4} - y^2}}$ **dydx**

$= \frac{1}{2} \int_0^2 x^2 \left[\sin^{-1}\left(\frac{y}{x^2/2}\right)\right]_0^{\frac{x^2}{2}}$ **dy**

$\therefore$ **I** $= \frac{1}{2} \int_0^2 x^2 \frac{\pi}{2}\, dx$

$= \frac{\pi}{4} \left[\frac{x^3}{3}\right]_0^2$

$\boxed{\therefore \text{ I } = \frac{2\pi}{3}}$

(b) Evaluate $\int\int\int x^2 dxdydz$ over the volume bounded by planes x=0,y=0 z=0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$ **[8]**

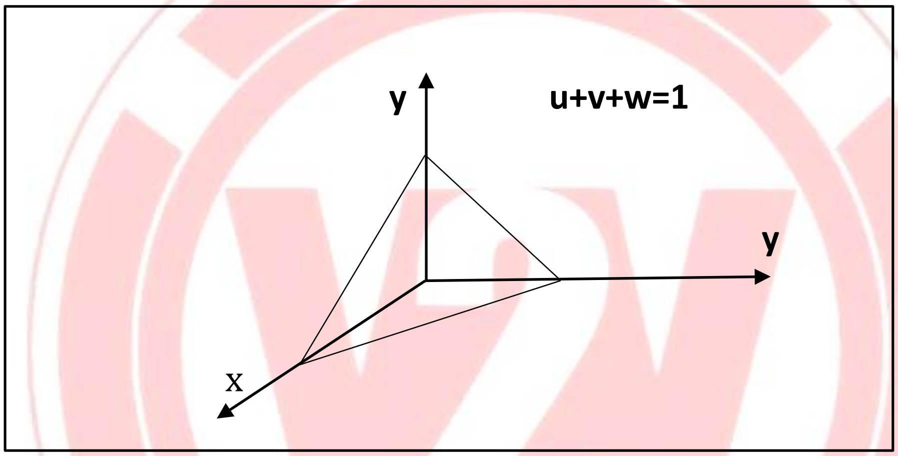
Ans : **Let V =** $\int\int\int x^2 dxdydz$

Region of integration is volume bounded by the planes x=0,y=0,z=0

And $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Put x = au , y = bv , z = cw

$\therefore$ dxdydz = abc du.dv.dw



The intersection of tetrahedron with all axes is : (1,0,0),(0,1,0),(0,0,1).

$$0 \le w \le (1-u-v)$$

$$0 \le v \le (1-u)$$

$$0 \le u \le 1$$

The volume required is given by ,

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\, a^2u^2\, dudvdw$$

$$= a^3bc \int_0^1 \int_0^{1-u} (1-u-v)\, u^2 dvdu$$

$$= a^3bc \int_0^1 u^2 \left[v - uv - \frac{v^2}{2} \right]_0^{1-u} du$$

$$= a^3bc\int_0^1 u^2[\,1-u-u+u^2-\frac{u^2(1-u)^2)}{2}\,]\,du$$

$$= a^3bc\,[\,\frac{u^3}{3}-\frac{u^4}{2}+\frac{u^5}{5}-\frac{1}{2}(\frac{u^3}{3}-\frac{1}{2}u^4+\frac{u^5}{5})\,]_0^1$$

$$\boxed{\therefore V = \frac{1}{60}(a^3bc)}$$

(c) Solve by method of variation of parameters : $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$ **[8]**

Ans : $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$

For complementary solution ,

$$f(D)=0$$

$$\therefore\ (D^2-6D+9)=0$$

Roots are : D = 3 , 3 Real roots but repeatative.

The complementary solution of given diff. eqn is ,

$$\boxed{\therefore\ y_c=(c_1+xc_2)e^{3x}}$$

For particular solution ,

By method of variation of parameters,

$$y_p=y_1p_1+y_2p_2$$

where $p_1=\int\frac{-y_2\,X}{w}\,dx$

$$p_2=\int\frac{y_1\,X}{w}\,dx$$

$$w=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix} = e^{6x}$$

$$p_1=\int\frac{-y_2\,X}{w}\,dx=\int\frac{xe_{3x}}{e^6}\cdot\frac{e^{3x}}{x^2}\,\text{dx} = \int\frac{-1}{x}\,dx=-\log x$$

$$p_2 = \int \frac{y_1 X}{w} dx = \int \frac{e^{3x}}{e^{6x}} . \frac{e^{3x}}{x^2} dx = \int \frac{1}{x^2} dx = \frac{-1}{x}$$

The particular integral of given diff. eqn is given by,

$$\boxed{\therefore y_p = -e^{3x} \log x - e^{3x} = -e^{3x}(\log x + 1)}$$

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c + y_p = (c_1 + x c_2)e^{3x} - e^{3x}(\log x + 1)}$$